Name: ______________________________

Teacher: ______________________________

Class: ______________________________

FORT STREET HIGH SCHOOL

2009

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 4: TRIAL HSC

Mathematics Extension 2

TIME ALLOWED: 3 HOURS

(PLUS 5 MINUTES READING TIME)

Outcomes Assessed	Questions	Marks
Determines the important features of graphs of a wide variety of functions, including conic sections	4, 5	
Applies appropriate algebraic techniques to complex numbers and polynomials	2, 3	
Applies further techniques of integration, such as slicing and cylindrical shells, integration by parts and recurrence formulae, to problems	1, 6	
Synthesises mathematical solutions to harder problems and communicates them in an appropriate form	7, 8	

Question	1	2	3	4	5	6	7	8	Total	%
Marks	/15	/15	/15	/15	/15	/15	/15	/15	/120	

Directions to candidates:

- Attempt all questions
- The marks allocated for each question are indicated
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used
- Each new question is to be started on a new page

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1. Start a NEW booklet (15 marks)

a) Show that $\int_2^4 \frac{dx}{x\sqrt{x-1}} = \frac{\pi}{6}$ **3**

b) Find the following indefinite integrals

i) $\int x^3 e^{-x}\, dx$ **3**

ii) $\int \frac{x^2-2x+6}{\left(x^2+4\right)\left(x-1\right)}\, dx$ **3**

iii) $\int \frac{2x+5}{\sqrt{x^2+4x+2}}\, dx$ **3**

iv) $\int \frac{dx}{2+\sin x}$ **3**

Question 2. Start a NEW booklet (15 marks)

a) The complex number ω is given by $-1+i\sqrt{3}$.

i) Show $\omega^2 = 2\overline{\omega}$ **1**

ii) Evaluate $|\omega|$ and $\arg\omega$ **2**

iii) Show that ω is a root of $\omega^3 - 8 = 0$ **2**

b) Sketch the region on the Argand diagram whose points z satisfy the inequalities $|z-\overline{z}| \le 4$ and $\frac{-\pi}{3} \le \arg z \le \frac{\pi}{3}$ **4**

c)

i) Prove that $\left(1+i\tan\theta\right)^n + \left(1-i\tan\theta\right)^n = 2\sec^n\left(\theta\right)\cos\left(n\theta\right)$ **3**

ii) Hence prove $\operatorname{Re}\left(\left(1+i\tan\frac{\pi}{8}\right)^8\right) = 64\left(12\sqrt{2}-17\right)$ **3**

Question 3. Start a NEW booklet (15 marks)

a) P and Q are points on the curve $y = x^4 + 4x^3$ where $x = \alpha$ and $x = \beta$ respectively. The line $y = mx + b$ is a tangent to the curve at both points P and Q.

i) By forming an expression for m when $x = \alpha$ and $x = \beta$, show that α and β are double roots of $x^4 + 4x^3 - mx - b = 0$. **2**

ii) Use the relationships between the roots and the coefficients of this equation to find the values of m and b. **3**

b) When a polynomial $P(x)$ is divided by $(x-3)$ the remainder is 5, and when it is divided by $(x-4)$ the remainder is 9.

Find the remainder when $P(x)$ is divided by $(x-4)(x-3)$. **3**

c) The equation $x^3 + kx + 2 = 0$ has roots α, β, γ.

i) Find an expression for $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ in terms of k. **2**

ii) Show that the value of $\alpha^3 + \beta^3 + \gamma^3$ is independent of k. **2**

iii) Find the monic equation with roots $\alpha^2 + \beta^2 + \gamma^2$ and coefficients in terms of k. **3**

Question 4. Start a NEW booklet (15 marks)

a) Consider the curves $\frac{x^2}{16}+\frac{y^2}{7}=1$ and $x^2-\frac{y^2}{8}=1$

i) Show that both curves have the same foci. **2**

ii) Find the equation of the circle through the points of intersection of the two curves. **4**

b) The point $P\left(x_o, y_0\right)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $a > b > 0$.

i) Write down the equations of the two asymptotes of the hyperbola. **1**

ii) Show that the acute angle α between the two asymptotes satisfies $\tan\alpha=\frac{2ab}{a^2-b^2}$ **2**

iii) If M and N are the feet of the perpendiculars drawn from P to the asymptotes, show that $MP.NP=\frac{a^2b^2}{a^2+b^2}$ **3**

iv) Hence show that the area of $\triangle PMN$ is $\frac{a^3b^3}{\left(a^2+b^2\right)^2}$ square units. **3**

Question 5. Start a NEW booklet (15 marks)

a) Consider the function $f\left(x\right)=2-\dfrac{4}{x^2+1}$.

i) Show that the function is even. **1**

ii) Find the coordinates of any points of intersection with the axes and the equations of any asymptotes of the graph $y=f\left(x\right)$. **2**

iii) Find the coordinates and nature of any stationary points of $y=f\left(x\right)$. **2**

iv) Sketch the graph of $y=f\left(x\right)$ showing all the above features. **2**

v) Draw separate one-third page sketches of the graphs of the following:

α. $y=\left|f\left(x\right)\right|$ **1**

β. $y=\left[f\left(x\right)\right]^2$ **2**

χ. $y^2=f\left(x\right)$ **2**

b) If y is a function of x which satisfies the relation $xy=ke^{\frac{y}{x}}$ where k is a constant, show that

$x\left(x-y\right)\dfrac{dy}{dx}+y\left(x+y\right)=0$ **3**

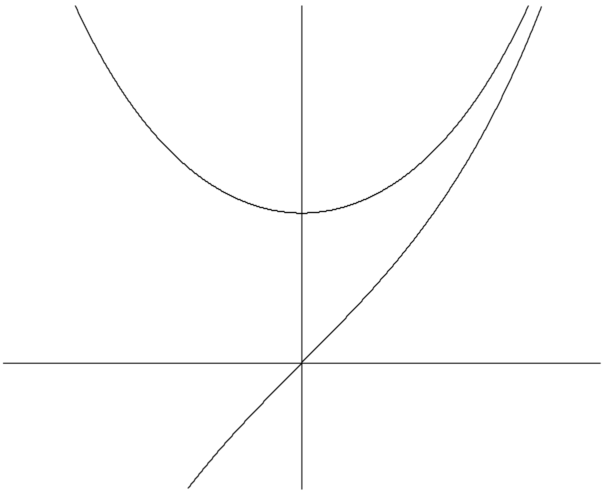
Question 6. Start a NEW booklet (15 marks)

a) The region between the curve $y = 4x - x^2$ and the x-axis is rotated about the line $x = 5$. Find the volume of the solid generated. **4**

b) The region bounded by the curves $g(x) = x^3$ and $f(x) = x^4$ is rotated about the x axis.

Using the method of slicing calculate the volume of the solid generated. **5**

c) The region between the curves $y = \dfrac{e^x + e^{-x}}{2}$ and $y = \dfrac{e^x - e^{-x}}{2}$, the y axis and the line $x = 1$ is rotated about the y axis.



i) Use the method of cylindrical shells to show that the volume of the solid generated is given by $V = 2\pi\int_0^1 xe^{-x}\ dx$ **4**

ii) Find the exact value of this volume. **2**

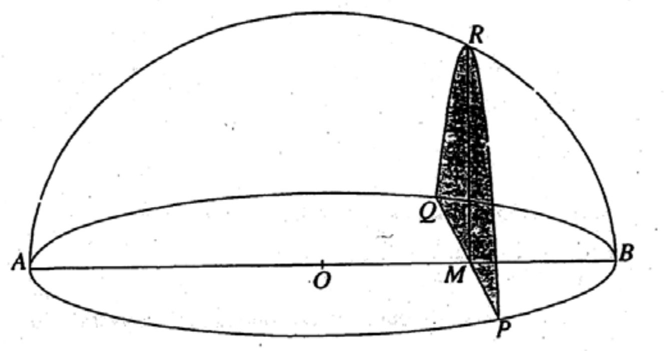
Question 7. Start a NEW booklet (15 marks)

a) A parabola passes through the three points $O(0,0)$, $A(a,h)$ and $B(-a,h)$, where a and h are positive real numbers.

i) Sketch the curve and find its equation. **1**

ii) Show that the area contained between the parabola and the line AB is two thirds of the area of the rectangle with vertices *A*, *B*, $M(a,0)$ and $N(-a,0)$. **3**

b)



In the diagram above, a tent has a circular base with centre *O* and radius *l*, and *AOB* is a of the base. The shaded area *PMQR* is a typical cross section of the tent perpendicular to *AB*, and meets *AB* at a point *M* distant *x* from *O*.

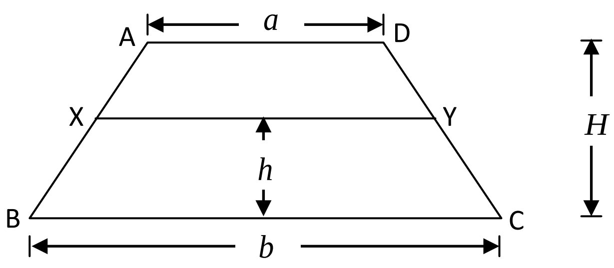
The curve *PRQ* is a parabola with axis *RM* and *QM* = *RM*.

i) Show that $MQ = \sqrt{l^2 - x^2}$ **1**

ii) Use part (a) to show that the shaded area *PMQR* is $\frac{4}{3}\left(l^2 - x^2\right)$ **1**

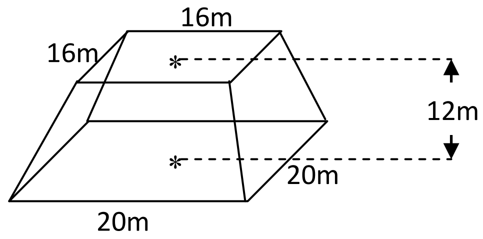
iii) Find the volume of the tent. **3**

c) ABCD is an isosceles trapezium of height H with $AB = DC$. The parallel sides AD and BC are of lengths a and b respectively. X and Y are points on AB and DC respectively such that XY is parallel to BC. The perpendicular distance between XY and BC is h.



i) Show that the length of XY is given by $XY = b - \frac{(b-a)h}{H}$ **3**

ii) The solid shown has a square base of 20m by 20m and a square top of 16m by 16m.



The top and base lie on two parallel planes. The four sides are isosceles trapeziums. The height of the solid is 12m. Find the volume of the solid by taking slices parallel to the base. **3**

Question 8. Start a NEW booklet (15 marks)

a)

i) Use the substitution $u = \frac{\pi}{4} - x$ to show that $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx$ **2**

ii) Hence find the exact value of $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$ **1**

b)

i) Find $\int \sin(7x)\sin(3x)\,dx$ **3**

ii) Find $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$ **3**

c) If $I_n = \int_0^1 \frac{1}{\left(x^2+1\right)^n}\,dx$

i) Prove $I_{n+1} = \frac{1}{2n}\left[2^{-n} + (2n-1)I_n\right]$ **4**

ii) Hence evaluate I_3 **2**

Question 1:

a) Show that $\int_2^4 \frac{dx}{x\sqrt{x-1}} = \frac{\pi}{6}$

Let :

$u = \sqrt{x-1}$

$du = \frac{dx}{2\sqrt{x-1}}$ $\quad x = 4, u = \sqrt{3}$, $\quad x = 2, u = 1$ [setup relationships❶]

$x = u^2 + 1$

Therefore:

$$\int_2^4 \frac{dx}{x\sqrt{x-1}}$$

$= \int_1^{\sqrt{3}} \frac{2du}{u^2+1}$ [correct substitutions (limits and expressions) ❶]

$$= 2\left[\tan^{-1} u\right]_1^{\sqrt{3}}$$

$= 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$ [correct integration & values ❶]

$$= \frac{\pi}{6}$$

b) Find the following indefinite integrals:

i) $\int x^3 e^{-x}\, dx$

Either,

$u = x^3 \qquad dv = e^{-x}$

$du = 3x^2 \qquad v = -e^{-x}$

so

$= -x^3e^{-x} + \int 3x^2 e^{-x} dx$ [setup and first integral correct ❶]

$= -x^3e^{-x} - 3x^2e^{-x} + 6\int xe^{-x} dx$

$= -x^3e^{-x} - 3x^2e^{-x} - 6xe^{x} + \int 6e^{-x} x dx$ [correct signs ❶]

$= -x^3e^{-x} - 3x^2e^{-x} - 6xe^{x} - 6e^{-x} + c$ [correct answer (incl +*c*) ❶]

or Tabular Form:

$u, du's$		$dv, dv's$	
x^3	[+]	e^{-x}	[correct expressions ❶]
$3x^2$	[-]	$-e^{-x}$	
$6x$	[+]	e^{-x}	
6	[-]	$-e^{-x}$	[correct signs ❶]
0		e^{-x}	

$= -x^3e^{-x} - 3x^2e^{-x} - 6xe^{x} - 6e^{-x} + c$ [correct answer (incl +*c*) ❶] *ii)* $\int \frac{x^2-2x+6}{(x^2+4)(x-1)}\,dx$

Partial fractions:

$\frac{x^2-2x+6}{(x^2+4)(x-1)} \equiv \frac{Ax+B}{(x^2+4)} + \frac{C}{(x-1)}$

$x^2-2x+6 \equiv (Ax+B)(x-1)+C(x^2+4)$

$x=1:\ 5=5C \Rightarrow C=1$

$x=0: 6=-B+4C \Rightarrow B=-2$ [correct values ❶]

$x=-1: 9=-2(-A+B)+4C \Rightarrow A=0$

$\therefore \int \frac{x^2-2x+6}{(x^2+4)(x-1)}\,dx$

$= \int \frac{-2}{(x^2+4)} + \frac{1}{(x-1)}\,dx$ [correct substitution ❶]

$= -\tan^{-1}\left(\frac{x}{2}\right) + \ln(x-1) + c$ [correct answer ❶]

iii) $\int \frac{2x+5}{\sqrt{x^2+4x+2}}\,dx$

$\int \frac{2x+4+1}{\sqrt{x^2+4x+2}}\,dx$

$= \int \frac{2x+4}{\sqrt{x^2+4x+2}} + \frac{1}{\sqrt{x^2+4x+2}}\,dx$ [correct split of integral ❶]

$= 2\sqrt{x^2+4x+2} + \int \frac{1}{\sqrt{(x+2)^2-2}}\,dx$ [correct root integral ❶]

$= 2\sqrt{x^2+4x+2} + \ln\left|(x+2)+\sqrt{(x+2)^2-2}\right| + c$

[correct log integral ❶]

iv) $\int \frac{dx}{2+\sin x}$

Using $t=\tan\left(\frac{x}{2}\right)$; $dx=\frac{2dt}{1+t^2}$; $\sin x=\frac{2t}{1+t^2}$, and hence

$\int \frac{dx}{2+\sin x}$

$= \int \frac{\frac{2dt}{1+t^2}}{2+\frac{2t}{1+t^2}}$ [correct t substitution ❶]

$= \int \frac{dt}{1+t+t^2}$

$$= \int \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2\left(t+\frac{1}{2}\right)}{\sqrt{3}}\right) + c$$ [correct integration ❶]

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}}\left(2\tan\frac{x}{2}+1\right)\right) + c$$ [in terms of x ❶]

Question 2:

a) The complex number ω is given by $-1+i\sqrt{3}$.

i) Show $\omega^2 = 2\overline{\omega}$.

$\omega^2 = \left(-1+i\sqrt{3}\right)^2$

$= 1 - 2i\sqrt{3} - 3$

$= -2 - 2i\sqrt{3}$ [correct working ❶]

$= 2\left(-1 - i\sqrt{3}\right)$

$= 2\overline{\omega}$

Mostly well done

ii) Evaluate $|\omega|$ and $\arg\omega$.

Noting $\omega = -1+i\sqrt{3}$ is in the 2nd quadrant:

$\arg\omega = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$ and $|\omega| = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2}$

$= \frac{2\pi}{3}$ [answer❶] $= 2$ [answer ❶]

Many did not consider the quadrant that $\arg\omega$ should be in

iii) Show that ω is a root of $\omega^3 - 8 = 0$.

$\omega^3 = \omega^2\omega$ [uses identity from i) ❶]

$= 2\overline{\omega}\omega$

$= 2|\omega|^2$

$= 8$ [answer ❶]

Hence ω is a root of $\omega^3 - 8 = 0$.

b) Sketch the region on the Argand diagram whose points

z satisy the inequalities $\left|z-\overline{z}\right|\le 4$ *and* $\frac{-\pi}{3}\le \arg\left(z\right)\le \frac{\pi}{3}$.

Letting $z=x+iy$:

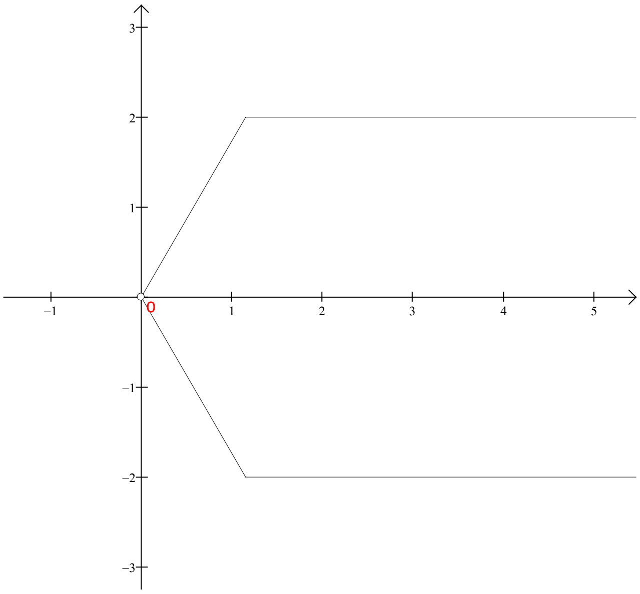
$$\left|z-\overline{z}\right|=\left|x+iy-\left(x-iy\right)\right| \qquad \text{Hence } 2\left|y\right|\le 4$$

$$=\left|2iy\right| \qquad \left|y\right|\le 2$$

$$=2\left|y\right| \qquad \Rightarrow -2\le y\le 2$$

Many did not use $z=x+iy$ correctly in the formula

Mod is equivalent to absolute value, and the -2 for the y limit was often missed.



correct y calculations ❶
correct args in diagram ❶
correct point at Origin ❶
correct region shaded ❶

c)

i) Prove that

$$\left(1+i\tan\theta\right)^n+\left(1-i\tan\theta\right)^n=2\sec^n\left(\theta\right)\cos\left(n\theta\right).$$

$$\left(1+i\tan\theta\right)^n+\left(1-i\tan\theta\right)^n$$

$$=\left(1+i\frac{\sin\theta}{\cos\theta}\right)^n+\left(1-i\frac{\sin\theta}{\cos\theta}\right)^n \text{ [correct substitution ❶]}$$

$$=\left(\frac{\cos\theta+i\sin\theta}{\cos\theta}\right)^n+\left(\frac{\cos\theta-i\sin\theta}{\cos\theta}\right)^n$$

$$=\frac{\cos\left(n\theta\right)+i\sin\left(n\theta\right)}{\cos^n\theta}+\frac{\cos\left(n\theta\right)-i\sin\left(n\theta\right)}{\cos^n\theta} \text{ [correct DeMoivre ❶]}$$

$$=\frac{2\cos\left(n\theta\right)}{\cos^n\theta}$$

$$=2\cos\left(n\theta\right)\sec^n\left(\theta\right) \text{ [correct resolution ❶]}$$

As reqd.

Mostly well done

ii) Hence prove $\text{Re}\left(\left(1+i\tan\frac{\pi}{8}\right)^8\right)=64\left(12\sqrt{2}-17\right)$

Noting $\text{Re}(z)=\frac{1}{2}\text{Re}(z+\overline{z})$,

$\text{Re}\left(\left(1+i\tan\frac{\pi}{8}\right)^8\right)$

$=\frac{1}{2}\text{Re}\left(\left(1+i\tan\frac{\pi}{8}\right)^8+\left(1-i\tan\frac{\pi}{8}\right)^8\right)$

then from i), with $n=8,\theta=\frac{\pi}{8}$

$=\frac{1}{2}.2\cos\left(8.\frac{\pi}{8}\right)\sec^8\left(\frac{\pi}{8}\right)$

$=-\sec^8\left(\frac{\pi}{8}\right)$ [correct resolution using i) ❶]

Now $\cos^2\left(\frac{\pi}{8}\right)=\frac{1}{2}\left(\cos\frac{\pi}{4}+1\right)$, using the $\cos 2\theta$ identity.

$=\frac{1+\sqrt{2}}{2\sqrt{2}}$ [correct from identity ❶]

Hence

$\cos^8\left(\frac{\pi}{8}\right)=\left(\frac{1+\sqrt{2}}{2\sqrt{2}}\right)^4$

$=\frac{1+4\sqrt{2}+12+8\sqrt{2}+4}{64}$

$=\frac{17+12\sqrt{2}}{64}$

$\therefore\sec^8\left(\frac{\pi}{8}\right)$

$=\frac{64}{17+12\sqrt{2}}\times\frac{17-12\sqrt{2}}{17-12\sqrt{2}}$

$=64\left(17-12\sqrt{2}\right)$ [correct resolution ❶]

Hence $\text{Re}\left(\left(1+i\tan\frac{\pi}{8}\right)^8\right)=-\sec^8\left(\frac{\pi}{8}\right)$ as reqd.

$=64\left(12\sqrt{2}-17\right)$

Many did not refer back to the previous part, thus making this part more complex and longer than it should have been.

Most did not connect the use of double angle formulae with the solutiojn to this problem, so were unable to complete it.

Question 3:

a) *P and Q are on the curve* $y = x^4 + 4x^3$ *where* $x = \alpha$ *and* $x = \beta$ *respectively. The line* $y = mx + b$ *is a tangent to the curve at both points P and Q.*

i) *By forming an expression for* m *when* $x = \alpha$ *and* $x = \beta$, *show that* α *and* β *are double roots of* $y = x^4 + 4x^3 - mx - b$.

With $f(x) = x^4 + 4x^3$, $f'(x) = 4x^3 + 12x^2$

At $x = \alpha$, $m_\alpha = 4\alpha^3 + 12\alpha^2$

At $x = \beta$, $m_\beta = 4\beta^3 + 12\beta^2$ [correct expressions for m_α, m_β ❶]

Then for $f(x) = x^4 + 4x^3 - mx - b$

$f'(x) = 4x^3 + 12x^2 - m$

To be a double root, $f'(\alpha) = 0$ and $f'(\beta) = 0$

But

$f'(\alpha) = 4\alpha^3 + 12\alpha^2 - m_\alpha$, and with $m_\alpha = 4\alpha^3 + 12\alpha^2$

$= 4\alpha^3 + 12\alpha^2 - \left(4\alpha^3 + 12\alpha^2\right)$

$= 0$

Similarly

$f'(\beta) = 4\beta^3 + 12\beta^2 - m_\beta$, and with $m_\beta = 4\beta^3 + 12\beta^2$

$= 4\beta^3 + 12\beta^2 - \left(4\beta^3 + 12\beta^2\right)$

$= 0$ [correct deduction for roots ❶]

Hence, α and β are double roots.

ii) *Use the relationships between the roots and co-efficients of this equation to find the values of* m *and* b.

Relationships with roots and co-efficients:

$\sum\alpha : \alpha + \alpha + \beta + \beta = -4 \Rightarrow \alpha + \beta = -2$ eqn ①

$\sum\alpha\beta : \alpha^2 + 2\alpha\beta + 2\alpha\beta + \beta^2 = 0 \Rightarrow (\alpha + \beta)^2 + 2\alpha\beta = 0$ eqn ②

$\sum\alpha\beta\chi : 2\alpha^2\beta + 2\alpha\beta^2 = m \Rightarrow 2\alpha\beta(\alpha + \beta) = m$ eqn ③

$\sum\alpha\beta\chi\delta : \alpha^2\beta^2 = -b \Rightarrow b = -\alpha^2\beta^2$ eqn ④

① in ② gives

$(-2)^2 + 2\alpha\beta = 0$

$\therefore \alpha\beta = -2$ [correct setup to give $\alpha\beta$ correct ❶]

In ④ gives $b = -4$ [correct b value ❶]

In ③ gives $m = 2.(-2)(-2)$

i.e. $m = 8$ [correct m value ❶]

b) When a polynomial $P(x)$ is divided by $(x-3)$ the remainder is 5, and when it is divided by $(x-4)$ the remainder is 9.
Find the remainder when $P(x)$ is divided by $(x-4)(x-3)$.

$P(3)=5 \Rightarrow P(x)=(x-3)Q_1(x)+5$

$P(4)=9 \Rightarrow P(x)=(x-4)Q_2(x)+9$

When $P(x)$ is divided by $(x-3)(x-4)$, the degree of the remainder is less than 2. i.e.

$R(x)=ax+b$

$\therefore P(x)=(x-3)(x-4)Q_3(x)+(ax+b)$ [set-up relationships ❶]

Thus when

$x=3: P(3)=5 \Rightarrow 5=3a+b$ … ①

$x=4: P(4)=9 \Rightarrow 9=4a+b$ … ② [equations formed ❶]

②-①: $\Rightarrow a=4$

Subst into ①: $12+b=5$

$b=-7$

Hence the remainder is $4x-7$. [remainder correct ❶]

c) The equation $x^3+kx+2=0$ has roots α, β, γ.

i) Find an expression for $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$ in terms of k.

Firstly, $\alpha+\beta+\gamma=0$, $\alpha\beta+\alpha\gamma+\beta\gamma=k$ and $\alpha\beta\gamma=-2$. Then:

$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma}$ [re-arranges expression correctly ❶]

$=-\frac{k}{2}$ [answer correct ❶]

ii) Show that the value of $\alpha^3+\beta^3+\gamma^3$ is independent of k.

Re-arranging the equation to $x^3=-kx-2$, and substituting the roots:

$\alpha^3=-k\alpha-2$

$\beta^3=-k\beta-2$

$\gamma^3=-k\gamma-2$

Hence, adding gives

$\alpha^3+\beta^3+\gamma^3=-k(\alpha+\beta+\gamma)-6$ [cubic relationship formed ❶]

$=-k(0)-6$

$=-6$ [value ❶]

Which is independent of k.

iii) Find the monic equation with roots $\alpha^2, \beta^2, \gamma^2$ (with coefficients in terms of k).

New roots are $y = \alpha^2, \beta^2, \gamma^2$, so $\alpha = \sqrt{y}$

Subst back in original equation gives:

$\left(\sqrt{y}\right)^3 + k\sqrt{y} + 2 = 0$

$y\sqrt{y} + k\sqrt{y} = -2$ [eqn with $\sqrt{y}$ formed ❶]

$\sqrt{y}\left(y + k\right) = -2$ and squaring both sides gives

$y\left(y + k\right)^2 = 4$ [squared eqn correct ❶]

$y\left(y^2 + 2ky + k^2\right) = 4$

$y^3 + 2ky^2 + k^2 y = 4$, so

$x^3 + 2kx^2 + k^2 x - 4 = 0$ [final eqn correct ❶]

is the monic equations with roots $\alpha^2, \beta^2, \gamma^2$.

Question 4:

a) Consider the curves $\frac{x^2}{16} + \frac{y^2}{7} = 1$ and $x^2 - \frac{y^2}{8} = 1$

i) Show that both curves have the same foci.

For the ellipse $\frac{x^2}{16} + \frac{y^2}{7} = 1$:

$a = 4$

$e = \sqrt{1 - \frac{7}{16}}$

$= \frac{3}{4}$

for the hyperbola $x^2 - \frac{y^2}{8} = 1$:

$a = 1$

$e = \sqrt{1 + \frac{8}{1}}$

$= 3$ [e's correct ❶]

Hence the foci are:

$\left(\pm ae, 0\right)$

$= \left(\pm 4 \times \frac{3}{4}, 0\right)$

$= \left(\pm 3, 0\right)$

$\left(\pm ae, 0\right)$

$= \left(\pm 1 \times 3, 0\right)$

$= \left(\pm 3, 0\right)$ [foci correct ❶]

ii) Find the equation of the circle through the points of intersection of the two curves.

① $8x^2 - y^2 = 8$ for the hyperbola.

② $7x^2 + 16y^2 = 128$ for the ellipse

$128x^2 - 16y^2 = 128$ (③=① $\times 16$) [eqn's correct ❶]

②+③: $135x^2 = 240$

$x^2 = \frac{16}{9}$

$\therefore x = \pm \frac{4}{3}$

[x value correct ❶]

then $y^2 = 8 \times \frac{16}{9} - 8$

$= \frac{56}{9}$

$\therefore y = \pm \frac{\sqrt{56}}{3}$

[y value correct ❶]

Eqn of circle through these points is given by $x^2 + y^2 = r^2$

i.e. $x^2 + y^2 = \frac{16}{9} + \frac{56}{9}$

or $x^2 + y^2 = 8$ [eqn correct ❶]

b) The point $P(x_o, y_0)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where a > b > 0.

i) Write down the equations of the two asymptotes of the hyperbola.

$y = \frac{b}{a}x \quad y = -\frac{b}{a}x$ [eqns correct ❶]

ii) Show that the acute angle α between the two asymptotes satisfies $\tan\alpha = \frac{2ab}{a^2 - b^2}$

Using $m_1 = \frac{b}{a}; m_2 = -\frac{b}{a}$ with $\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ gives:

$$\tan\alpha = \left|\frac{\frac{b}{a} - \frac{-b}{a}}{1 + \left(\frac{b}{a}\right)\left(\frac{-b}{a}\right)}\right| \text{ [subst correct ❶]}$$

$$= \left|\frac{\frac{2b}{a}}{1 - \frac{b^2}{a^2}}\right|$$

$$= \frac{2ab}{a^2 - b^2} \text{ since } 0 < b < a \text{ [algebra correct ❶]}$$

iii) If M and N are the feet of the perpendiculars drawn from P to the asymptotes, show that $MP.NP = \frac{a^2b^2}{a^2 + b^2}$

Asymptotes in general form: $bx - ay = 0; bx + ay = 0$, hence:

$MP = \frac{|ax + by + c|}{\sqrt{a^2 + b^2}}$ and $NP = \frac{|ax + by + c|}{\sqrt{a^2 + b^2}}$ [*MP; NP* correct ❶]

$= \frac{|bx_0 - ay_0|}{\sqrt{a^2 + b^2}} \qquad = \frac{|bx_0 + ay_0|}{\sqrt{a^2 + b^2}}$

$$MP.NP = \frac{|bx_0 - ay_0|}{\sqrt{a^2 + b^2}} \cdot \frac{|bx_0 + ay_0|}{\sqrt{a^2 + b^2}}$$

$$= \frac{|b^2x_0^2 - a^2y_0^2|}{a^2 + b^2} \qquad \text{[expression correct ❶]}$$

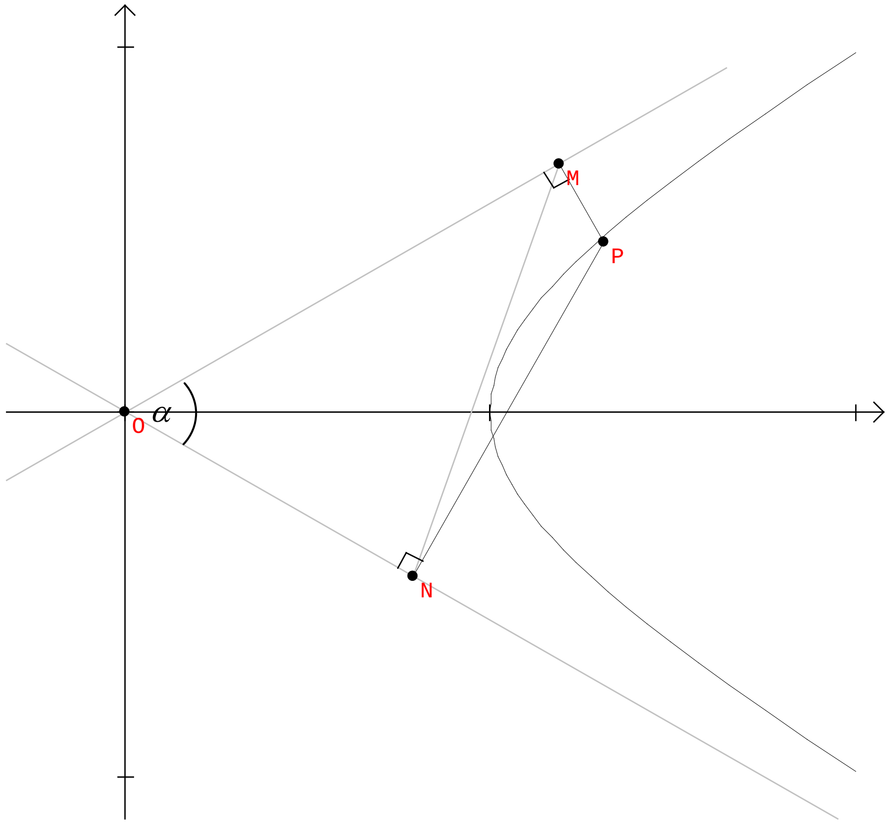
Since $P(x_o, y_0)$ lies on the hyperbola, $\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1$

Or $b^2x_0^2 - a^2y_0^2 = a^2b^2$ [uses P on hyperbola correctly ❶]

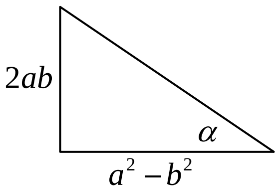
$$MP.NP = \frac{|a^2b^2|}{a^2+b^2}$$

$$= \frac{a^2b^2}{a^2+b^2} \text{ as reqd.}$$

iv) Hence show that the area of $\triangle PMN$ *is* $\dfrac{a^3b^3}{\left(a^2+b^2\right)^2}$ *square units.*



With $\tan(\alpha) = \dfrac{2ab}{a^2-b^2}$, then by Pythagoras:



$$h^2 = \left(a^2-b^2\right)^2 + (2ab)^2$$

$$= a^4 - 2a^2b^2 + b^4 + 4a^2b^2$$

$$= \left(a^2+b^2\right)^2$$

$h = a^2 + b^2$ [finds h ❶]

Now, $\angle MPN = \pi - \alpha$, hence $\sin(\pi-\alpha) = \sin\alpha$.

So $\sin\alpha = \dfrac{2ab}{a^2+b^2}$, and using [finds $\sin\alpha$ ❶]

$A = \dfrac{1}{2}ab\sin C$ in $\triangle PMN$ gives:

$$A_{PMN} = \frac{1}{2}\cdot\frac{a^2b^2}{a^2+b^2}\cdot\frac{2ab}{a^2+b^2}$$

$$= \frac{a^3b^3}{\left(a^2+b^2\right)^2}$$ as reqd [algebra correct ❶]

Question 5:

a) Consider the function $f(x)=2-\dfrac{4}{x^2+1}$.

i) Show that the function is even.

$$f(-x)=2-\frac{4}{(-x)^2+1}$$
$$=2-\frac{4}{x^2+1}$$
$$=f(x) \quad \text{[tests correctly ❶]}$$

$\therefore f(x)$ is even.

Show all steps – specially what happens to the minus signs in odd/even function proofs!

ii) Find the coordinates of any points of intersection with the axes and the equations of any asymptotes of the graph $y=f(x)$.

Intercepts: [both intercepts correct ❶]

When $x=0$,

$$f(0)=2-\frac{4}{1}$$
$$=-2$$

and when $y=0$

$$0=2-\frac{4}{x^2+1}$$
$$\frac{2}{x^2+1}=1$$
$$2=x^2+1$$
$$x^2=1$$
$$x=\pm 1$$

Generally well done

Asymptotes:

As $x\to\pm\infty, \dfrac{4}{x^2+1}\to 0, \therefore y\to 2$

$\therefore y=2$ is a horizontal asymptote. [asymptote correct ❶]

Often the reason for this asymptote was omitted, and $y=2$ was stated with no reasoning shown

iii) Find the coordinates and nature of any stationary points of $y=f(x)$.

$\dfrac{dy}{dx}=\dfrac{8x}{(x^2+1)^2}$, so $\dfrac{dy}{dx}=0 \Rightarrow x=0$ and $f(0)=-2$

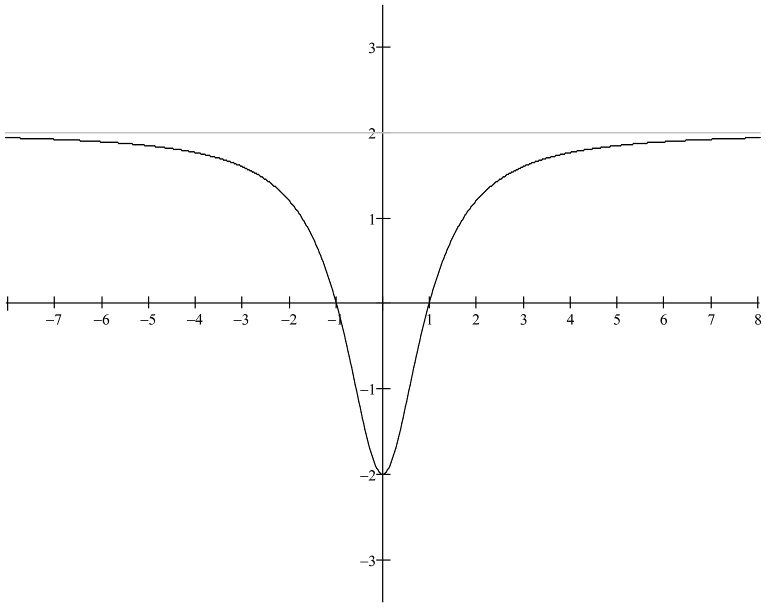
x	0^-	0	0^+
y'	<0	$=0$	>0
	\	_	/

i.e. a min tp. [tests correctly ❶]

Hence $(0,-2)$ is a minimum. pt. [finds pt ❶]

As is far too usual, the test for type of stat pt was omitted.

iv) *Sketch the graph of* $y = f(x)$ *showing all the above features.*

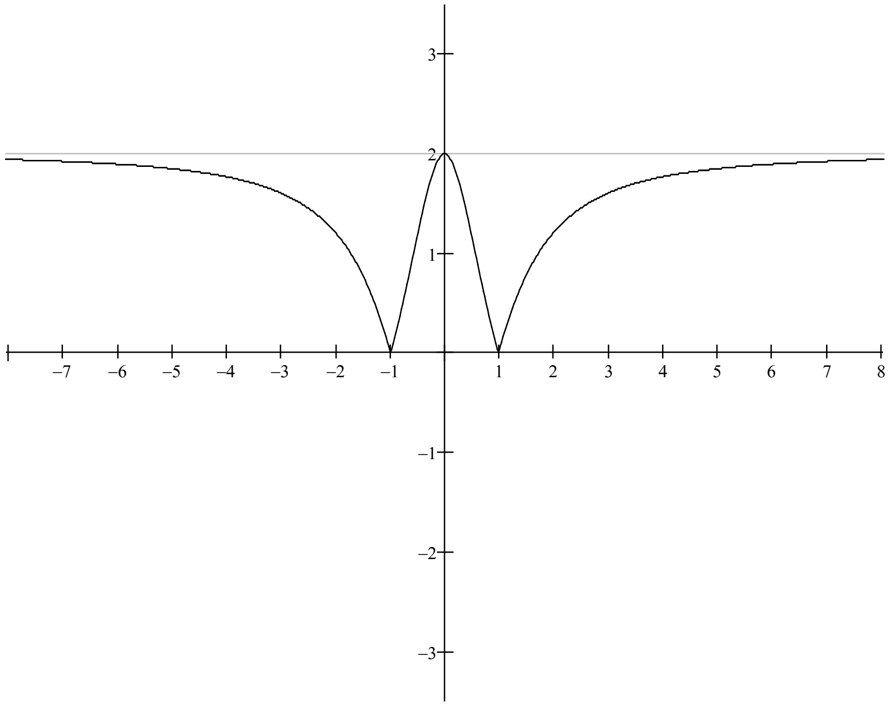


Generally well done

[intercepts/asymptotes correct ❶]; [shape correct ❶]

v) *Draw separate one-third page sketches of the graphs of the following:*

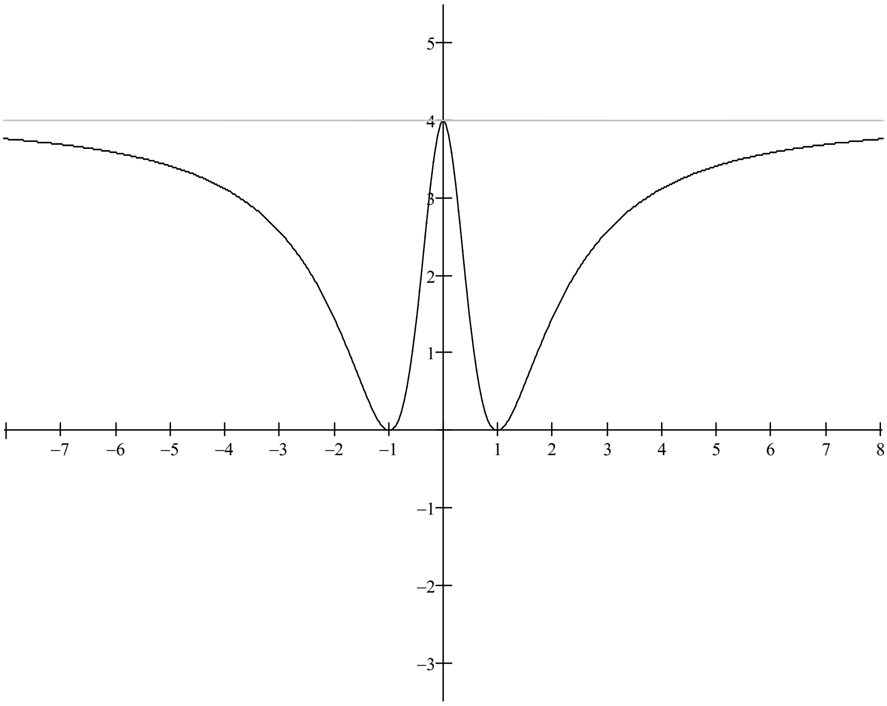
α. $y = |f(x)|$



Generally well done

[shape correct ❶]

β. $y = \left[f(x) \right]^2$

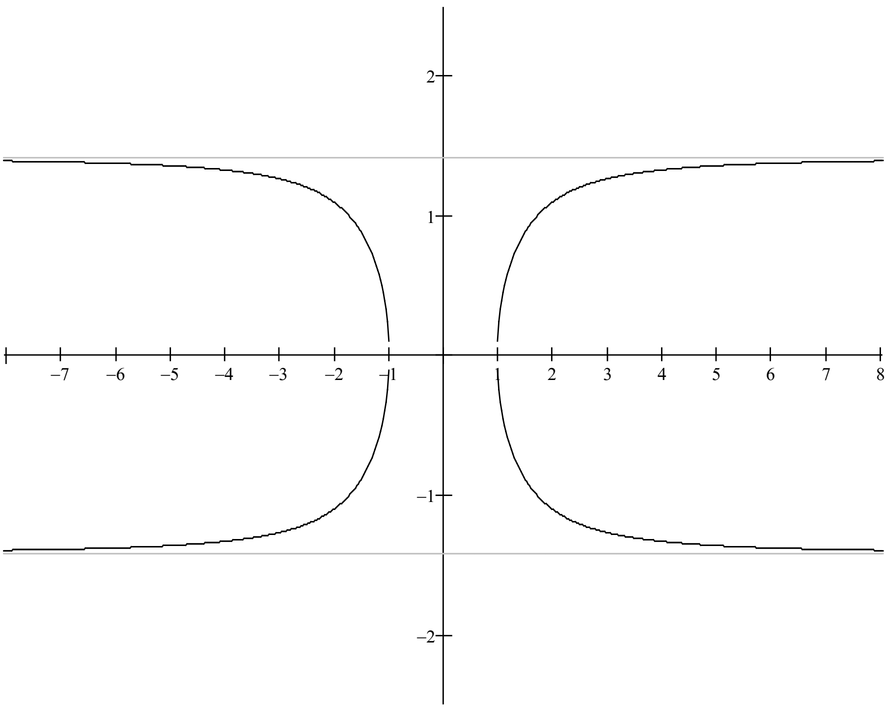


Asymptote value often not squared.

Roots become parabolic in nature – often drawn as a cusp indicating confusion with absolute value.

[intercepts/asymptotes correct ❶]; [shape correct ❶]

χ. $y^2 = f(x)$



Asymptotes are square roots of the original values here – often left out.

$y^2 = f(x)$ is a reflection in the x-axis…many omitted the lower half of the graph.

Some included the region $(-1 < x < 1)$ where the original graph is negative!

[intercepts/asymptotes correct ❶]; [shape correct ❶] *b) If y is a function of x which satisfies the relation*

$xy = ke^{\frac{y}{x}}$ *where k is a constant, show that*

$x(x-y)\frac{dy}{dx} + y(x+y) = 0$

Many tried to differentiate implicitly with limited success.

$$xy = ke^{\frac{y}{x}}$$

$$\frac{xy}{k} = e^{\frac{y}{x}}$$

Taking logs of both sides:

$$\frac{y}{x} = \ln\left(\frac{xy}{k}\right)$$

$$= \ln x + \ln y + \ln k$$ [algebra correct ❶]

δ.

Differentiating both sides:

$$\frac{x\frac{dy}{dx} - y}{x^2} = \frac{1}{x} + \frac{\frac{dy}{dx}}{y}$$

$$\frac{\frac{dy}{dx}}{x} - \frac{y}{x^2} = \frac{1}{x} + \frac{\frac{dy}{dx}}{y}$$

$$\frac{\frac{dy}{dx}}{x} - \frac{\frac{dy}{dx}}{y} = \frac{1}{x} + \frac{y}{x^2}$$

$$\frac{dy}{dx}\left(\frac{1}{x} - \frac{1}{y}\right) = \frac{x+y}{x^2}$$ [algebra correct ❶]

$$\frac{dy}{dx}\left(\frac{y-x}{xy}\right) = \frac{x+y}{x^2}$$

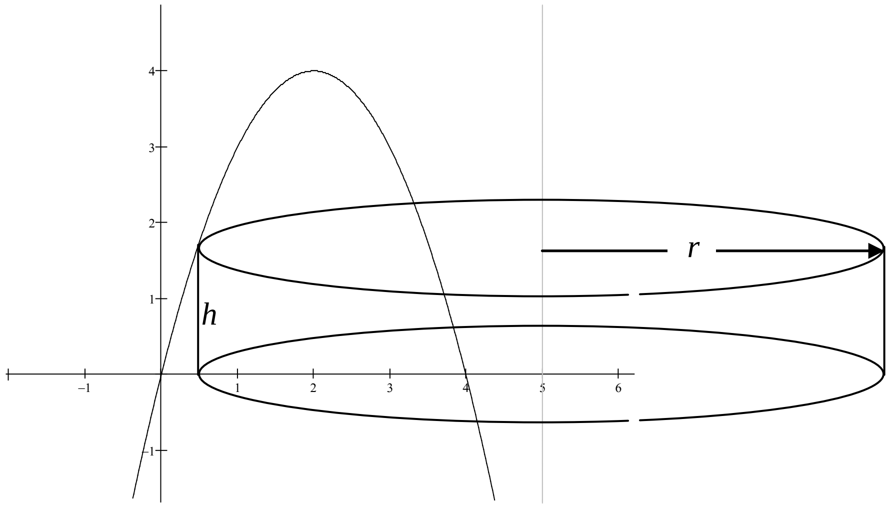
$$x^2(y-x)\frac{dy}{dx} = xy(x+y)$$ [algebra correct ❶]

$$0 = x(x-y)\frac{dy}{dx} + y(x+y)$$

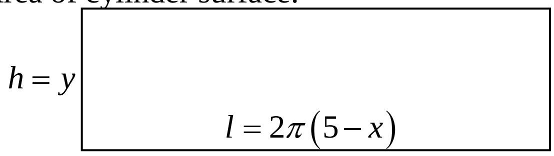
Question 6.

a) *The region between the curve* $y = 4x - x^2$ *and the* x-*axis is rotated about the line* $x = 5$. *Find the volume of the solid generated.*

By cylindrical shells: $h = y$ and $r = 5 - x$



Area of cylinder surface:



[diagrams of curve/cylinder and δA correct ❶]

So $\delta A = 2\pi(5-x)y$ and

$$\delta V = 2\pi(5-x)y\ \delta x$$

$$= 2\pi(5-x)\left(4x-x^2\right)\delta x \quad [\delta V \text{ correct ❶}]$$

Then $V = \sum \delta V$

$$= \lim_{\delta x \to 0} \sum_{x=0}^{4} 2\pi(5-x)\left(4x-x^2\right)\delta x$$

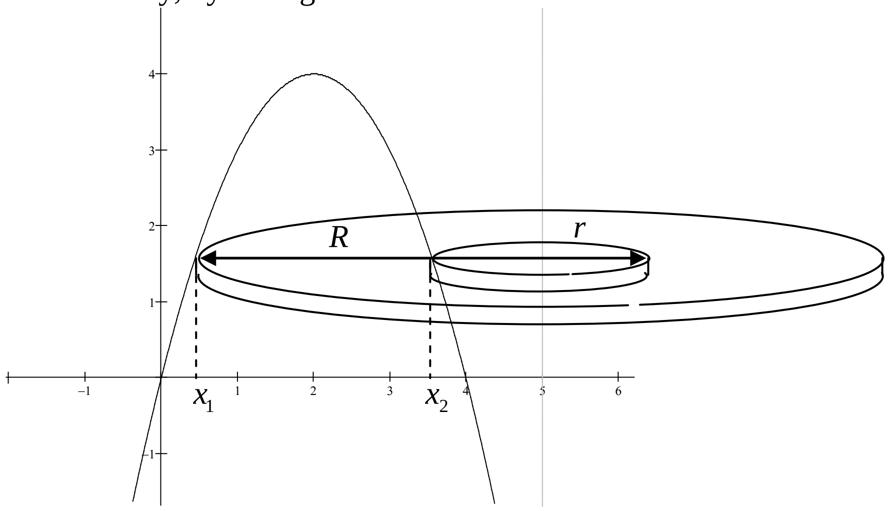
Hence $V = \int_0^4 2\pi\left(20x - 9x^2 + x^3\right)dx$ [V correct ❶]

$$= 2\pi\left[10x^2 - 3x^3 + \frac{1}{4}x^4\right]_0^4$$

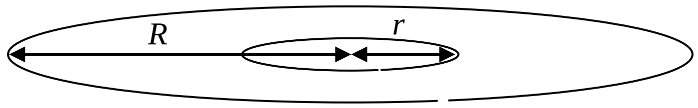
$$= 2\pi(160 - 192 + 64)$$

$$= 64\pi$$ [answer correct ❶]

Alternatively, by slicing:



For a slice: $\delta A = \pi R^2 - \pi r^2$, where $R = 5 - x_1; r = 5 - x_2$



x_1, x_2 are the solutions to $x^2 - 4x + y = 0$, considering y a constant. i.e.

$$x_i = \frac{4 \pm \sqrt{16 - 4y}}{2}$$
$$= 2 \pm 2\sqrt{4 - y}$$

[diagrams of curve/slice and δA correct ❶]

Hence $R = 5 - \left(2 + 2\sqrt{4 - y}\right)$; $r = 5 - \left(2 - 2\sqrt{4 - y}\right)$, giving:

$$\delta A = \pi\left(R - r\right)\left(R + r\right)$$
$$= \pi\left(-2\sqrt{4 - y}\right)\left(6\right)$$
$$= -12\pi\sqrt{4 - y} \quad \text{, and}$$

$\delta V = -12\pi\sqrt{4 - y}\,\delta y$ [δV correct ❶]

Then $V = \sum \delta V$

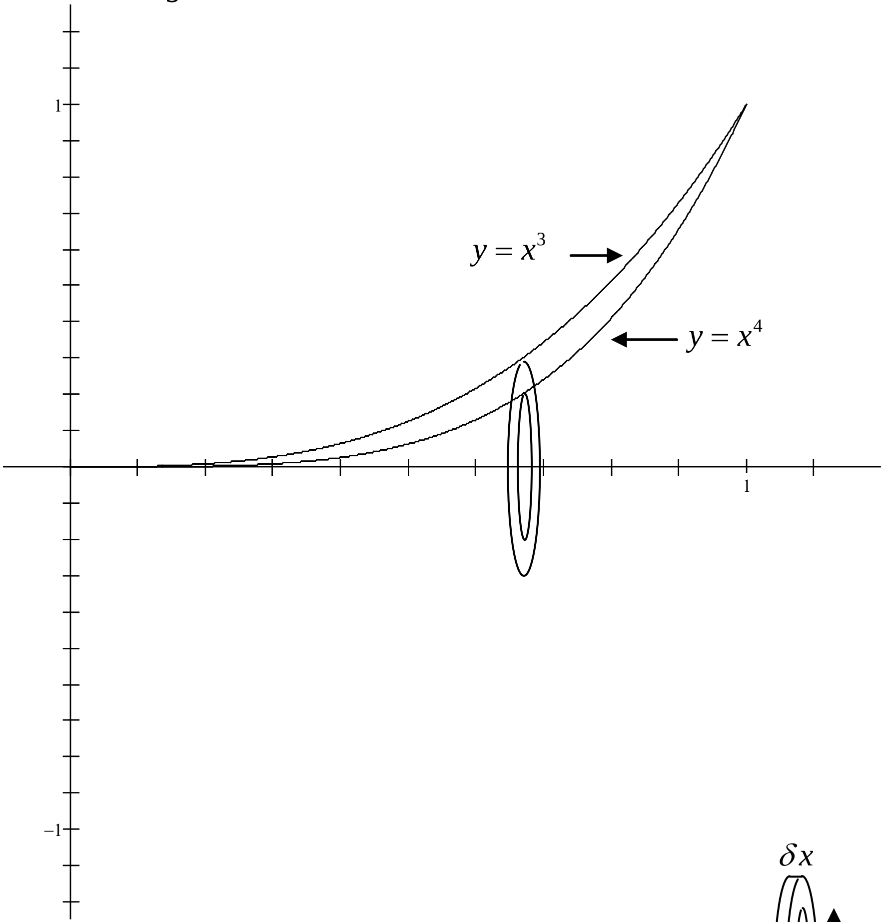
$$= \lim_{\delta y \to 0} \sum_{x=0}^{4} -12\pi\sqrt{4 - y}\,\delta y$$

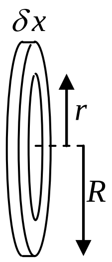
Hence $V = \int_0^4 -12\pi\sqrt{4 - y}\,dy$ [V correct ❶]

$$= -12\pi\left[\frac{2}{3}\left(4 - y\right)^{\frac{3}{2}}\right]_0^4$$
$$= -12\pi\left(\frac{-2}{3}.8\right)$$
$$= 64\pi$$

[answer correct ❶]

b) The region bounded by the curves $g(x) = x^3$ *and* $f(x) = x^4$ *is rotated about the x axis. Using the method of slicing calculate the volume of the solid generated.*





Inner radius: $r = x^4$

Outer radius: $R = x^3$

Area of Annulus: $\delta A = \pi\left(R^2 - r^2\right)$

[diagrams of curve/slice and δA correct ❶]

Volume of a slice: $\delta V = \pi\left(R - r\right)\left(R + r\right)\delta x$ [δV correct ❶]

Then

$$V = \sum \delta V$$

$$= \lim_{\delta x \to 0} \sum_{x=0}^{1} \left(x^3 + x^4\right)\left(x^3 - x^4\right)\delta x$$ [uses correct notation ❶]

Hence $V = \int_0^1 \pi\left(x^3 + x^4\right)\left(x^3 - x^4\right)dx$ [V correct ❶]

$$= \pi \int_0^1 x^6 - x^8 \, dx$$

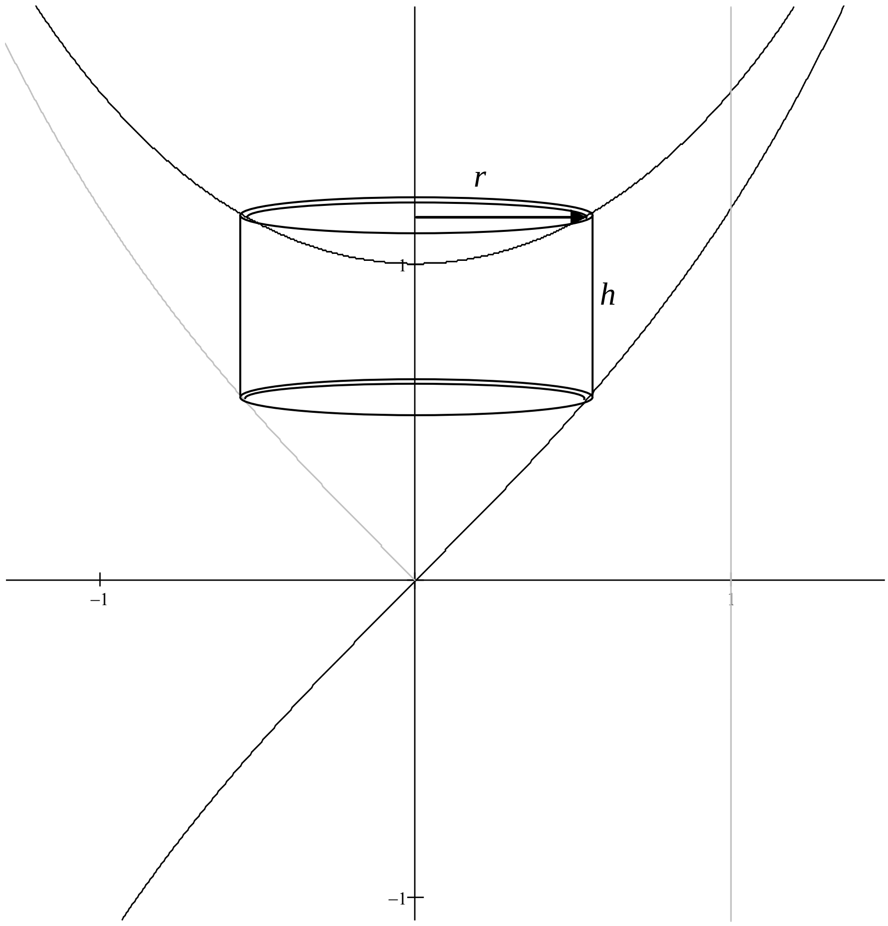
$$= \pi \left[\frac{x^7}{7} - \frac{x^9}{9} \right]_0^1$$

$$= \pi \left(\frac{1}{7} - \frac{1}{9} \right)$$

$$= \frac{2\pi}{63}$$ [answer correct ❶]

c) The region between the curves $y = \frac{e^x + e^{-x}}{2}$ *and* $y = \frac{e^x - e^{-x}}{2}$*, the y axis and the line x = 1 is rotated about the y axis.*

i) Use the method of cylindrical shells to show that the volume of the solid generated is given by $V = 2\pi \int_0^1 x e^{-x} \; dx$



Area of cylinder surface:

$h = y_1 - y_2$

$l = 2\pi x$

[diagrams of curve/cylinder correct ❶]

Height: $h = \frac{e^{x}+e^{-x}}{2} - \frac{e^{x}-e^{-x}}{2}$

$= \frac{2e^{-x}}{2}$

$= e^{-x}$ [expression for h correct ❶]

Hence annulus has:

$\delta A = 2\pi x e^{-x}$, and [δA expressions correct ❶]

$\delta V = 2\pi x e^{-x}\,\delta x$ [δV expressions correct ❶]

Then

$V = \sum \delta V$

$= \lim_{\delta x \to 0} \sum_{x=0}^{1} 2\pi x e^{-x}\,\delta x$

Hence $V = 2\pi \int_0^1 x e^{-x}\,dx$ as reqd.

ii) Find the exact value of this volume.

$V = 2\pi \int_0^1 x e^{-x}\,dx$, then by parts: $u = x \quad dv = e^{-x}$

$du = dx \quad v = -e^{-x}$

Hence

$v = 2\pi \left\{ \left[-xe^{-x}\right]_0^1 - \int_0^1 -e^{-x}\,dx \right\}$ [by parts correct ❶]

$= 2\pi \left\{ \left(-e^{-1} - 0\right) + \left[-e^{-x}\right]_0^1 \right\}$

$= 2\pi \left[-e^{-1} + \left(-e^{-1} + e^{0}\right) \right]$

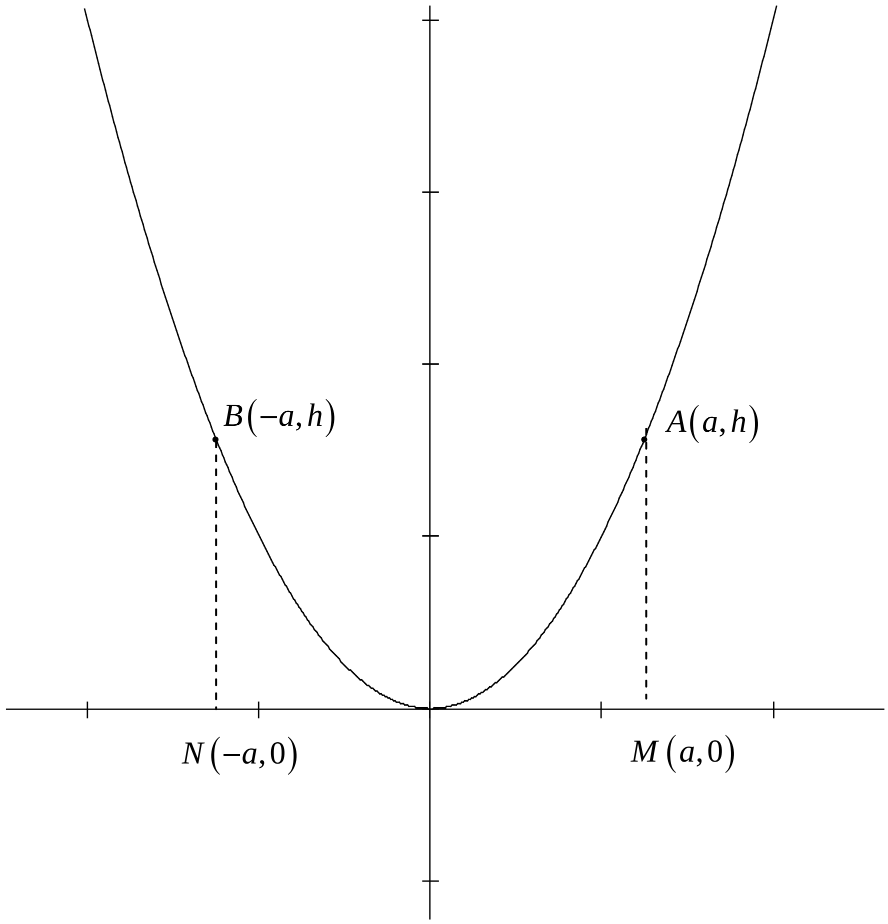
$= 2\pi \left(-2e^{-1} + 1\right)$

$= 2\pi \left(1 - \frac{2}{e}\right) units^3$ [answer correct ❶]

<u>Question 7.</u>

a) A parabola passes through the three points $O(0,0)$, $A(a,h)$ *and* $B(-a,h)$, *where a and h are positive real numbers.*

i) Sketch the curve and find its equation.



Since the parabola passes through (0,0) and is symmetrical about the y-axis, its equation is of the form $y = kx^2$.

Passing through $A(a,h)$: $h = ka^2$ so $k = \frac{h}{a^2}$

Hence its equation is $y = \frac{h}{a^2}x^2$. [graph and eqn correct ❶]

ii) *Show that the area contained between the parabola and the line AB is two thirds of the area of the rectangle with vertices A, B, $M(a,0)$ and $N(-a,0)$.*

Area of $ABNM = 2ah$

So area between parabola and line AB is given by:

$A = 2ah - \int_{-a}^{a} \frac{hx^2}{a^2}\,dx$ [area formula correct ❶]

$$= 2ah - \int_{-a}^{a} \frac{hx^2}{a^2}\,dx$$

$$= 2ah - \frac{2h}{a^2}\int_{0}^{a} x^2\,dx$$

$$= 2ah - \frac{2h}{a^2}\left[\frac{x^3}{3}\right]_0^a$$

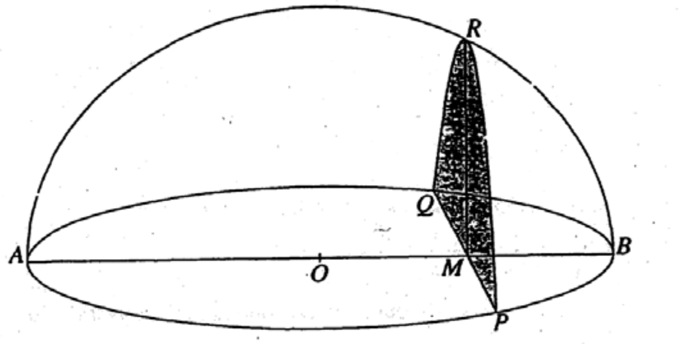
$$= 2ah - \frac{2ah}{3}$$

$= \frac{4ah}{3}$ [integration correct ❶]

$= \frac{2}{3} \times 2ah$

$= \frac{2}{3} \times ABNM$ [reasoning to show relationship correct ❶]

b)



In the diagram above, a tent has a circular base with centre O and radius l, and AOB is a diameter of the base. The shaded area PMQR is a typical cross section of the tent perpendicular to AB, and meets AB at a point M distant x from O.
The curve PRQ is a parabola with axis RM and QM = RM.

i) Show that $MQ = \sqrt{l^2 - x^2}$ **1**

$QM^2 = OQ^2 - OM^2$

$= l^2 - x^2$ [shows links correctly ❶]

$QM = \sqrt{l^2 - x^2}$

ii) Use part (a) to show that the shaded area $PMQR$ is $\frac{4}{3}\left(l^2 - x^2\right)$ **1**

Area $PMQR = \frac{2}{3}\left(QP \times RM\right)$ (from (a))

$A = \frac{2}{3}\left(2\sqrt{l^2 - x^2}\sqrt{l^2 - x^2}\right)$ [shows links correctly ❶]

$= \frac{4}{3}\left(l^2 - x^2\right)$

iii) Find the volume of the tent. **3**

$V = \int_{-l}^{l} \frac{4}{3}\left(l^2 - x^2\right)dx$ [set up V correctly ❶]

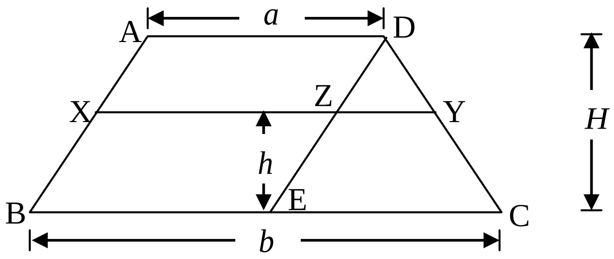
$= \frac{8}{3}\left[l^2 x - \frac{1}{3}x^3\right]_0^l$ [integration & subst correct ❶]

$= \frac{16}{9}l^3$ [answer correct ❶]

c) *ABCD is an isosceles trapezium of height H with $AB = DC$. The parallel sides AD and BC are of lengths a and b respectively. X and Y are points on AB and DC respectively such that XY is parallel to BC. The perpendicular distance between XY and BC is h.*

i) *Show that the length of XY is given by*

$$XY = b - \frac{(b-a)h}{H}$$



Constructing $DE \parallel AB$, then $\Delta DZY \,|||\, \Delta DEC$ (equiangular).
[construction and similarity ❶]
Hence:

$$\frac{ZY}{b-a} = \frac{H-h}{H}$$

$$ZY = \frac{(b-a)(H-h)}{H}$$ [relationship correct ❶]

$$XY = a + \frac{(b-a)(H-h)}{H}$$

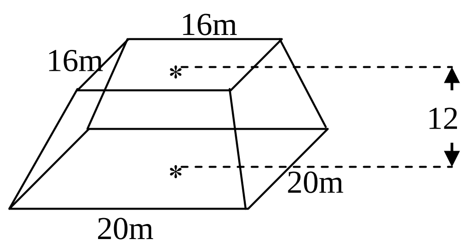
$$= \frac{aH + (b-a)H - (b-a)h}{H}$$

$$= \frac{aH + bH - aH - bh + ah}{H}$$

$$= \frac{bH - (b-a)h}{H}$$ [algebra correct ❶]

$$XY = b - \frac{(b-a)h}{H}$$

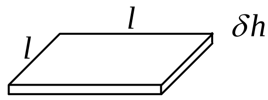
ii) *The solid shown has a square base of 20m by 20m and a square top of 16m by 16m.*



The top and base lie on two parallel planes. The four sides are isosceles trapeziums. The height of the solid is 12m. Find the volume of the solid by taking slices parallel to the base.

From *i)* above:

$l = 20 - \frac{(20-16)h}{12}$, so



$\delta A = \left(20 - \frac{(20-16)h}{12}\right)^2$ [diagram of slice and δA correct ❶]

and hence

$$\delta V = \left(20 - \frac{(20-16)h}{12}\right)^2 \delta h$$
$$= \left(20 - \frac{h}{3}\right)^2 \delta h$$ [δV correct ❶]

$\therefore$ Volume of solid:

$$V = \lim_{\delta h \to 0} \sum_0^{12} \left(20 - \frac{h}{3}\right)^2 \delta h$$
$$= \int_0^{12} \left(20 - \frac{h}{3}\right)^2 dh$$
$$= \left[-3\frac{\left(20 - \frac{h}{3}\right)^3}{3}\right]_0^{12}$$
$$= 3904\, m^3$$ [answer correct ❶]

<u>Question 8.</u>

a)

i) *Use the substitution* $u = \frac{\pi}{4} - x$ *to show*

that $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx$

When $u = \frac{\pi}{4} - x$, $du = -dx$ and $x = u + \frac{\pi}{4}$, and $x = 0 \Rightarrow u = \frac{\pi}{4}$

and $x = \frac{\pi}{4} \Rightarrow u = 0$; hence: [correct use of $(a-x)$ ❶]

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_{\frac{\pi}{4}}^0 \ln\left(1+\tan\left(\frac{\pi}{4}-u\right)\right) \times -du$$
$$= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan u}{1+\tan u}\right)du$$
$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{1+\tan u + 1 - \tan u}{1+\tan u}\right)du$$ [correct ❶]

Many did not show the limit changes.

Many did not change the variable all in one line.

Some did not recognise $\tan\frac{\pi}{4} = 1$

$$\therefore \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx \text{ as reqd.}$$

ii) Hence find the exact value of

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$$

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(2-(1+\tan x)\right)dx$$

Many missed this simplification.

$$= \int_0^{\frac{\pi}{4}} \ln 2\,dx - \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$$

$$\therefore 2\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln 2\,dx$$

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln 2\,dx$$

Some did not divide by 2.

$$= \left[\frac{x\ln 2}{2}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{8}\ln 2$$ [answer correct ❶]

b)

i) Find $\int \sin(7x)\sin(3x)\,dx$ ***3***

Noting $\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$

$$\sin 7x \sin 3x = \frac{1}{2}\left[\cos(7-3)x - \cos(7+3)x\right]$$

$$= \frac{1}{2}\left[\cos 4x - \cos 10x\right]$$

This identity not well used or known.

[correct use of trig identity ❶]

$$\therefore \int \sin 7x \sin 3x\,dx = \frac{1}{2}\int \cos 4x - \cos 10x\,dx$$

$$= \frac{1}{2}\left(\frac{1}{4}\sin 4x - \frac{1}{10}\sin 10x\right) + c$$ [working ❶]

Careless signs!

$$= \frac{1}{8}\sin 4x - \frac{1}{20}\sin 10x + c$$

[answer correct ❶]

ii) *Find* $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\,dx$$

Use of $\int_0^a f(a-x)dx=\int_0^a f(x)dx$ very poor – this is a standard method!

$$=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$ [expression correct ❶]

Hence:

$$2\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx+\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$=\int_0^{\frac{\pi}{2}} 1\,dx$$

$$=\left[x\right]_0^{\frac{\pi}{2}}$$

$$=\frac{\pi}{2}$$ [int correct ❶]

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx=\frac{\pi}{4}$$ [answer correct ❶]

c) *If* $I_n=\int_0^1 \frac{1}{\left(x^2+1\right)^n}\,dx$

i) *Prove* $I_{n+1}=\frac{1}{2n}\left[2^{-n}+\left(2n-1\right)I_n\right]$

Let $u=\left(x^2+1\right)^{-n}$, then $du=-n\left(x^2+1\right)^{-n-1}.2x\,dx$ and

$$=\frac{-2nx}{\left(x^2+1\right)^{n+1}}\,dx$$

$dv=1$; $v=x$. [parts correct ❶]

By parts components poorly done.

Then, by parts:

$$I_n=\left[x\left(x^2+1\right)^{-n}\right]_0^1-\int_0^1 x.\frac{-2nx}{\left(x^2+1\right)^{n+1}}\,dx$$

$$= 2^{-n} + 2n\int_0^1 \frac{x^2}{\left(x^2+1\right)^{n+1}}\,dx$$ [initial integration correct ❶]

$$= 2^{-n} + 2n\int_0^1 \frac{x^2+1-1}{\left(x^2+1\right)^{n+1}}\,dx$$

Another standard method that students must be familiar with – many were not.

$$= 2^{-n} + 2n\int_0^1 \frac{x^2+1}{\left(x^2+1\right)^{n+1}}\,dx - 2n\int_0^1 \frac{1}{\left(x^2+1\right)^{n+1}}\,dx$$

$$= 2^{-n} + 2n\int_0^1 \frac{1}{\left(x^2+1\right)^{n}}\,dx - 2n\int_0^1 \frac{1}{\left(x^2+1\right)^{n+1}}\,dx$$ [parts correct ❶]

$$= 2^{-n} + 2nI_n - 2nI_{n+1}$$

Thus

$$I_n = 2^{-n} + 2nI_n - 2nI_{n+1}$$

$$2nI_{n+1} = 2^{-n} + (2n-1)I_n$$ [algebra correct ❶]

Hence , $I_{n+1} = \frac{1}{2n}\left[2^{-n} + (2n-1)I_n\right]$ as reqd,

ii) Hence evaluate I_3

Need I_1, so

$$I_1 = \int_0^1 \frac{1}{x^2+1}\,dx$$

$$= \left[\tan^{-1} x\right]_0^1$$

$$= \frac{\pi}{4}$$ [I_1 correct ❶]

Many tried to evaluate I_0. Only do this if I_1 does not evaluate!

$$I_3 = \frac{1}{4}\left[2^{-2} + 3I_2\right]$$

Many errors here - $I_3 = I_{n+1}$ implies $n = 2$, which many missed.

$$= \frac{1}{16} + \frac{3}{4}I_2$$

$$= \frac{1}{16} + \frac{3}{4}.\frac{1}{2}\left[2^{-1} + I_2\right]$$

$$= \frac{1}{16} + \frac{3}{16} + \frac{3}{8}I_1$$

$$= \frac{1}{4} + \frac{3}{8}.\frac{\pi}{4}$$

$$= \frac{8+3\pi}{32}$$ [answer correct ❶]